Math 22
Quiz 5
Fri Nov 2, 2012

NAME YOU ASKED TO BE CALLED IN CLASS: ____________________

SCORE: _____ / 35 POINTS

1. **You may use the result of exercise 26 in section 4.4 without proving it.**
2. **You may NOT use the results of example 4.2.3 in section 4.2 unless you write formal proofs of them.**
3. **You may use the property that all integers are either even or odd,**
   AND the property that consecutive integers have opposite parity.

Prove by mathematical induction that $5^n + 10 < 6^n$ for all integers $n \geq 2$. **SCORE: _____ / 15 PTS**

There are two possible solutions for the inductive step, depending on which side of the inequality you started from.

BASIS STEP:

2 $5^2 + 10 = 26 < 36 = 6^2$

INDUCTIVE STEP:

2 Suppose $5^k + 10 < 6^k$ for some particular but arbitrary 1 integer $k \geq 2$ 1

[Need to prove: $5^{k+1} + 10 < 6^{k+1}$]

ONLY GRADE AGAINST 1 SOL'N

SOLUTION 1:

1 $6^{k+1} = 6 \cdot 6^k > 6(5^k + 10) = 6 \cdot 5^k + 60 > 6 \cdot 5^k + 10 > 5 \cdot 5^k + 10 = 5^{k+1} + 10$ 3 3 1

SOLUTION 2:

1 $5^{k+1} + 10 = 5 \cdot 5^k + 10 = 5(5^k + 10) - 40 < 5 \cdot 6^k - 40 < 5 \cdot 6^k < 6 \cdot 6^k = 6^{k+1}$ 3 3 1

1 So, by mathematical induction, $5^n + 10 < 6^n$ for all integers $n \geq 2$.

MANY OTHER SOLUTIONS POSSIBLE – COME TALK TO ME

Consider the statement "The square root of a positive irrational number is irrational." **SCORE: _____ / 5 PTS**

[a] Write **ONLY** the first complete sentence of a proof by contradiction. (The answer is much more than just 2 words.)

Suppose not, that is, suppose there is a positive irrational number whose square root is rational. 2

[b] Write **ONLY** the first and last complete sentences of a proof by contraposition. (QED is not a sentence.)

FIRST: Let x be a particular but arbitrary positive number such that $\sqrt{x}$ is rational. 2

LAST: Therefore, x is rational. 1

One of the following statements is true and one is false. **SCORE: _____ / 15 PTS**
Write a formal proof for the true statement, and show that the false statement is false.

[a] For all integers n, if $3 \mid (n^2 - 7)$, then $n \bmod 3 = 1$ or $n \bmod 3 = 2$.
[b] The product of a rational number and an irrational number is irrational.

[a] is true. **There are two possible solutions, depending on whether you used contraposition or contradiction.**

ONLY GRADE AGAINST 1 SOLUTION

SOLUTION 1:
PROOF BY CONTRAPOSITION:

CONTRAPOSITIVE: For all integers n, if $n \bmod 3 \neq 1$ and $n \bmod 3 \neq 2$, then $3 \nmid (n^2 - 7)$.

Let n be a particular but arbitrary integer such that $n \bmod 3 \neq 1$ and $n \bmod 3 \neq 2$. 1 POINT EACH
So, $n \bmod 3 = 0$ by QRT.
So, $n = 3q$ for some $q \in Z$ by definition of mod.
So, $n^2 - 7 = 9q^2 - 7 = 3(3q^2 - 3) + 2$ where $3q^2 - 3 \in Z$ by closure of Z under $\times$ and $-$.
So, $(n^2 - 7) \bmod 3 = 2$ by definition of mod.
So, $3 \nmid (n^2 - 7)$ by exercise 26 in section 4.4.
So, by contraposiion, for all integers n, if $3 \mid (n^2 - 7)$, then $n \bmod 3 = 1$ or $n \bmod 3 = 2$.

SOLUTION 2:
PROOF BY CONTRADICTION:

Suppose not, that is, suppose there is an integer n such that $3 \mid (n^2 - 7)$ and $n \bmod 3 \neq 1$ and $n \bmod 3 \neq 2$.
So, $n \bmod 3 = 0$ by QRT.
So, $n = 3q$ for some $q \in Z$ by definition of mod.
So, $n^2 - 7 = 9q^2 - 7 = 3(3q^2 - 3) + 2$ where $3q^2 - 3 \in Z$ by closure of Z under $\times$ and $-$.
So, $(n^2 - 7) \bmod 3 = 2$ by definition of mod.
So, $3 \nmid (n^2 - 7)$ by exercise 26 in section 4.4. 1 POINT EACH
But $3 \mid (n^2 - 7)$ (CONTRADICTION). $\frac{1}{2}$ ← EXCEPT →
So, by contradiction, for all integers n, if $3 \mid (n^2 - 7)$, then $n \bmod 3 = 1$ or $n \bmod 3 = 2$ $\frac{1}{2}$

[b] is false. 0 is rational and $\sqrt{2}$ is irrational, but $0 \times \sqrt{2} = 0$ is rational.

3